Computers and Mathematics with Applications 62 (2011) 3987–3996

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Classification of traveling wave solutions to the Vakhnenko equations

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ARTICLE INFO

Article history:
Received 28 February 2011
Received in revised form 17 September 2011
Accepted 26 September 2011

Keywords:
Traveling wave solution
Vakhnenko equation
Complete discrimination system for polynomial

ABSTRACT

The classification of all single traveling wave solutions to the Vakhnenko equation and its generalization are obtained by means of the complete discrimination system for the polynomial method.

1. Introduction

In the past several years, the classification of traveling wave solutions to some nonlinear differential equations has been studied extensively by the complete discrimination system for the polynomial method proposed by Liu [1–6]. Liu's method is so powerful that more nonlinear equations can be dealt with. For example, Wang and Li [7] used Liu's method and other methods to study the single solitary and multi-solitary solutions to some nonlinear equations. Yang [8] gave the classification of envelope solutions to the SD equation. In the present paper, we consider the following Vakhnenko equation [9] and its generalization:

$$u_{tx} + u_x^2 + uu_{xx} + u = f(u), \tag{1}$$

where

$$\text{(i) } f(u) = 0, \tag{2}$$

$$\text{(ii) } f(u) = \varepsilon u^2. \tag{3}$$

The Vakhnenko equation governs the propagation of waves in a relaxing medium [10]. In Refs. [9,11], some exact traveling wave solutions have been obtained. Recently, Li et al. [12] have applied the (ω/g)-expansion method to give some exact solutions to the Vakhnenko equation. Wu et al. [13] have used the Homotopy analysis method to give some approximation solutions to the Vakhnenko equation. Li [14] has used the trial function method to solve the Vakhnenko equation. Mo [15] has obtained some approximate solutions to several generalized Vakhnenko equations. There exist many papers discussing aspects of the Vakhnenko equation. For example, Morrison, Vakhnenko and Parkes have studied N loop solitons [16], Parkes has invested the stability of the Vakhnenko equation [9], Vakhnenko and Parkes [17] have also invested an interesting connection of the DP equation [18] with the Vakhnenko equation, and so on. In particular, in Refs. [17,18], the solutions have been illustrated in figures.

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doi:10.1016/j.camwa.2011.09.060

In his book [14], Li reduced the Vakhnenko equation to an integral form

$$-c\phi''' - c(\phi')^2 + \phi' = 0, \tag{4}$$

and thought that it was difficult to integrate (4) directly, where $u(x,t) = \phi_X(X,T)$, $\xi = X - cT + \xi_0$, T, X are two variables with $x = T + \phi(X,T) + x_0$, $t = X$, c and ξ_0 are two arbitrary constants. Li [14] obtained an exact solution

$$u(x,t) = \frac{3}{2}a^2 \sec\mathrm{h}^2\left(\frac{a}{2}\eta\right), \quad x - a^2 t = 3a\tan\mathrm{h}\left(\frac{a}{2}\eta\right) - a^2\eta + x_0^*, \tag{5}$$

where a and x_0^* are two arbitrary constants, by assuming that

$$\phi = \frac{B\exp a\xi}{1+\exp a\xi}, \tag{6}$$

where a and B are constants to be determined.

We must point out that Eq. (4) can be direct reduced to an integral form by a result given in Refs. [6]. In this paper, we study the traveling wave solution to the Vakhnenko equation and its generalization, and give the classifications of the single traveling wave solutions to the Eqs. (2) and (3). Among those, many new solutions are given.

2. Classifications

Under the traveling wave transformation $u = u(\xi)$, $\xi = hx + wt$, the generalized Vakhnenko equation becomes

$$u'' + \frac{h}{\omega+uh}(u')^2 + \frac{u-f(u)}{h(\omega+uh)} = 0. \tag{7}$$

From Ref. [6], the general solution of the above ODE is given by

$$\pm(\xi-\xi_0) = \int \frac{\mathrm{d}u}{\sqrt{\exp\left(-2\int\frac{h}{\omega+uh}\mathrm{d}u\right)\left[c - 2\int\frac{u-f(u)}{h(\omega+uh)}\exp\left(2\int\frac{h}{\omega+uh}\mathrm{d}u\right)\mathrm{d}u\right]}}. \tag{8}$$

According to the formula, we can give the corresponding solutions to the Vakhnenko equation and its generalization.

(i) $f(u) = 0$. The general solution becomes

$$\pm\frac{\sqrt{2}(\xi-\xi_0)}{\sqrt{3}h} = \int\sqrt{\frac{\left[-\left(u+\frac{\omega}{h}\right)\right]^2}{\left[-\left(u+\frac{\omega}{h}\right)\right]^3 + a_2\left[-\left(u+\frac{\omega}{h}\right)\right]^2 + a_1\left[-\left(u+\frac{\omega}{h}\right)\right] + a_0}}\mathrm{d}u, \tag{9}$$

where

$$a_2 = \frac{3\omega}{2h}, \qquad a_1 = 0, \qquad a_0 = \frac{3c}{2} - \frac{\omega^3}{2h^3}. \tag{10}$$

Take the change of the variable

$$v = -\left(u + \frac{\omega}{h}\right). \tag{11}$$

Then we have

$$\pm\frac{\sqrt{2}(\xi-\xi_0)}{\sqrt{3}h} = \int\sqrt{\frac{v^2}{v^3+a_2v^2+a_1v+a_0}}\mathrm{d}v. \tag{12}$$

According to the above integral, the classification of all solutions to the Vakhnenko equation is given as follows.

Case 1. $a_0 = 0$. We get the corresponding solutions

$$u = \frac{\omega}{2h} - \frac{(\xi-\xi_0)^2}{6h^2}. \tag{13}$$

Case 2. $a_0 \neq 0$. We denote $F(v) = v^3 + a_2v^2 + a_1v + a_0$. Its complete discrimination system is given by

$$\Delta = -27\left(\frac{2a_2^3}{27} + a_0 - \frac{a_1a_2}{3}\right)^2 - 4\left(a_1 - \frac{a_2^2}{3}\right)^3,$$

$$D_1 = a_1 - \frac{a_2^2}{3}. \tag{14}$$

Correspondingly, there are the following four cases to be discussed.

Case 2.1. $\Delta = 0, D_1 < 0$. We denote $F(v) = (v-\alpha)^2(v-\beta), \alpha \neq \beta, v > \beta$. Take the change of the variable

$$v = \beta + m^2, \tag{15}$$

and hence its inverse transformation is

$$v - \beta = m^2. \tag{16}$$

Therefore we obtain

$$\pm \frac{\sqrt{2}(\xi - \xi_0)}{\sqrt{3}h} = 2\sqrt{-u - \frac{\omega}{h} - \beta} + \frac{2\alpha}{\sqrt{\beta - \alpha}} \arctan \frac{\sqrt{-u - \frac{\omega}{h} - \beta}}{\sqrt{\beta - \alpha}}, \quad (\beta > \alpha), \tag{17}$$

and

$$\pm \frac{\sqrt{2}(\xi - \xi_0)}{\sqrt{3}h} = 2\sqrt{-u - \frac{\omega}{h} - \beta} + \frac{\alpha}{\sqrt{\alpha - \beta}} \ln \left| \frac{\sqrt{-u - \frac{\omega}{h} - \beta} - \sqrt{\alpha - \beta}}{\sqrt{-u - \frac{\omega}{h} - \beta} + \sqrt{\alpha - \beta}} \right|, \quad (\beta < \alpha). \tag{18}$$

Case 2.2. $\Delta = 0, D_1 = 0$. We denote $F(v) = (v-\alpha)^3,\ v > \alpha$. The solutions are given by

$$\pm \frac{\sqrt{2}(\xi - \xi_0)}{\sqrt{3}h} = 2\sqrt{-u - \frac{\omega}{h} - \alpha} - \frac{2\alpha}{\sqrt{-u - \frac{\omega}{h} - \alpha}}. \tag{19}$$

Case 2.3. $\Delta > 0, D_1 < 0$. We denote $F(v) = (v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$, and suppose $\alpha_1 > \alpha_2 > \alpha_3$. By assuming that

$$v = \frac{a \sin^2 \varphi + b}{c \sin^2 \varphi + d}, \quad \left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right), \tag{20}$$

when $v > \alpha_1$, we have

$$\pm \frac{\sqrt{2}(\xi - \xi_0)}{\sqrt{3}h} = \frac{2\delta}{\gamma} \left\{ \alpha_1 F(\varphi, k) - (\alpha_1 - \alpha_3) E(\varphi, k) + (\alpha_1 - \alpha_3) \tan \varphi \sqrt{1 - k^2 \sin^2 \varphi} \right\}, \tag{21}$$

where

$$a = -\alpha_2, \quad b = \alpha_1, \quad c = -1, \quad d = 1, \quad \delta = \alpha_1 - \alpha_2, \quad \gamma = (\alpha_1 - \alpha_2)\sqrt{\alpha_1 - \alpha_3},$$
$$k^2 = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3},$$
$$F(\varphi, k) = \int_0^{\sin \varphi} \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}}, \qquad E(\varphi, k) = \int_0^{\sin \varphi} \sqrt{\frac{1-k^2\eta^2}{1-\eta^2}} d\eta, \tag{22}$$

when $\alpha_2 > v > \alpha_3$, we have

$$\pm \frac{\sqrt{2}(\xi - \xi_0)}{\sqrt{3}h} = \frac{2\delta}{\gamma} \{\alpha_1 F(\varphi, k) - (\alpha_1 - \alpha_3) E(\varphi, k)\}, \tag{23}$$

where

$$a = \alpha_2 - \alpha_3, \quad b = \alpha_3, \quad c = 0, \quad d = 1, \quad \delta = \alpha_2 - \alpha_3,$$
$$\gamma = (\alpha_2 - \alpha_3)\sqrt{\alpha_1 - \alpha_3}, \quad k^2 = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}. \tag{24}$$

Case 2.4. $\Delta < 0$. We denote $F(v) = (v-\alpha_1)[(v-r)^2 + s^2]$, $(v > \alpha_1)$, where r and α_1 are real numbers, $s > 0$. By assuming that

$$v = \frac{a \cos \varphi + b}{c \cos \varphi + d}, \quad (0 \leqslant \varphi \leqslant \pi), \tag{25}$$

we have

$$\pm \frac{\sqrt{2}\gamma(\xi - \xi_0)}{\sqrt{3}(a+b)^2 h} = \frac{1 + \cos \varphi}{\sin \varphi} \sqrt{1 - k^2 \sin^2 \varphi} - \frac{b}{a+b} F(\varphi, k) + E(\varphi, k), \tag{26}$$

where

$$\tan 2\theta = \frac{s}{\alpha_1 - r}, \quad \left(0 < \theta < \frac{\pi}{2}\right) . \gamma^2 = \frac{4s^3}{\sin^3 2\theta},$$
$$a = -r + s \tan \theta, \quad b = r + s \cot \theta, \quad c = -1, \quad d = 1, \quad k = |\sin \theta|. \tag{27}$$

(ii) $f(u)=\varepsilon u^2$. The general solution becomes

$$\pm\frac{\sqrt{\epsilon\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}=\int\sqrt{\frac{\left(u+\frac{\omega}{h}\right)^2}{\epsilon\left[\left(u+\frac{\omega}{h}\right)^4+b_3\left(u+\frac{\omega}{h}\right)^3+b_2\left(u+\frac{\omega}{h}\right)^2+b_1\left(u+\frac{\omega}{h}\right)+b_0\right]}}\mathrm{d}u, \tag{28}$$

where

$$\begin{aligned}&\epsilon=\pm1,\qquad b_3=-\frac{8\varepsilon\omega+4h}{3h\varepsilon},\qquad b_2=\frac{2\varepsilon\omega^2+2h\omega}{h^2\varepsilon},\qquad b_1=0,\\&b_0=\frac{2c}{\varepsilon}-\frac{\varepsilon\omega^4+2h\omega^3}{3h^4\varepsilon}.\end{aligned} \tag{29}$$

If $\varepsilon>0$, we take $\epsilon=+1$; if $\varepsilon<0$, we take $\epsilon=-1$. We give the classifications of all solutions to the integral as follows.

Case 1. $b_0=0$. We denote $\Delta={b_3}^2-4b_2$. There are the following three cases to be discussed.

Case 1.1. $\Delta=0$. When $\epsilon=+1$, we get the corresponding solutions

$$u=\pm\exp\left(\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)+\frac{\omega}{3h}+\frac{2}{3\varepsilon}. \tag{30}$$

Case 1.2. $\Delta<0$. When $\epsilon=+1$, we get the corresponding solutions

$$u=\pm\frac{1}{2}\exp\left(\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)\mp\frac{\varepsilon^2\omega^2+\varepsilon\omega h-2h^2}{9\varepsilon^2h^2}\exp\left(\mp\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)+\frac{\varepsilon\omega+2h}{3h\varepsilon}. \tag{31}$$

Case 1.3. $\Delta>0$. When $\epsilon=+1$, we get the corresponding solutions

$$u=\pm\frac{1}{2}\exp\left(\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)\mp\frac{\varepsilon^2\omega^2+\varepsilon\omega h-2h^2}{9\varepsilon^2h^2}\exp\left(\mp\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)+\frac{\varepsilon\omega+2h}{3h\varepsilon}. \tag{32}$$

When $\epsilon=-1$, we have

$$u=\frac{\pm\sqrt{4h^2-2\varepsilon^2\omega^2-2\varepsilon\omega h}\sin\left(\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}\right)+\varepsilon\omega+2h}{3h\varepsilon}. \tag{33}$$

Case 2. $b_0\neq0$. We denote

$$F(u)=\left(u+\frac{\omega}{h}\right)^4+b_3\left(u+\frac{\omega}{h}\right)^3+b_2\left(u+\frac{\omega}{h}\right)^2+b_0. \tag{34}$$

Furthermore, we have

$$F(u)=(u+t)^4+p(u+t)^2+q(u+t)+r, \tag{35}$$

where

$$\begin{aligned}&t=\frac{\varepsilon\omega-h}{3h\varepsilon},\qquad p=-\frac{2\omega\varepsilon h+2\varepsilon^2\omega^2+2h^2}{3h^2\varepsilon^2},\qquad q=\frac{8\varepsilon^3\omega^3+12h\varepsilon^2\omega^2-12h^2\varepsilon\omega-8h^3}{27h^3\varepsilon^3},\\&r=\frac{2c}{\varepsilon}-\frac{\varepsilon^4\omega^4+2h\varepsilon^3\omega^3-6h^2\varepsilon^2\omega^2+2h^3\varepsilon\omega+h^4}{27h^4\varepsilon^4}.\end{aligned} \tag{36}$$

We take the change of variable

$$w=u+t. \tag{37}$$

Then Eq. (28) becomes

$$\pm\frac{\sqrt{\epsilon\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}=\int\sqrt{\frac{\left(w+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)^2}{\epsilon(w^4+pw^2+qw+r)}}\mathrm{d}w. \tag{38}$$

We denote

$$F(w)=w^4+pw^2+qw+r, \tag{39}$$

and write its complete discrimination system as follows

$$\begin{aligned}&D_1=4,\qquad D_2=-p,\qquad D_3=8rp-2p^3-9q^2,\\&D_4=4p^4r-p^3q^2+36prq^2-32r^2p^2-\frac{27}{4}q^4+64r^3,\qquad E_2=9q^2-32pr.\end{aligned} \tag{40}$$

Correspondingly, there are the following nine cases to be discussed.

Case 2.1. $D_4 = 0, D_3 = 0, D_2 < 0$. We have

$$F(w) = ((w - l_1)^2 + s_1^2)^2, \tag{41}$$

where l_1, s_1 are real numbers, $s_1 > 0$. When $\epsilon = +1$, we have

$$\pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} = \frac{1}{2} \ln \left[\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - l_1 \right)^2 + s_1^2 \right] + \frac{l_1 + \frac{2\varepsilon\omega + h}{3h\varepsilon}}{s_1} \arctan \frac{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - l_1}{s_1}. \tag{42}$$

Case 2.2. $D_4 = 0, D_3 = 0, D_2 = 0$. When $\epsilon = +1$, we have

$$F(w) = w^4, \tag{43}$$

and hence we have

$$\pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} = \ln \left| u + \frac{\varepsilon\omega - h}{3h\varepsilon} \right| - \frac{2\varepsilon\omega + h}{3h\varepsilon u + \varepsilon\omega - h}. \tag{44}$$

Case 2.3. $D_4 = 0, D_3 = 0, D_2 > 0, E_2 = 0$. We have

$$F(w) = (w - \alpha)^2 (w - \beta)^2, \tag{45}$$

where α, β are real numbers, $\alpha > \beta$. If $\epsilon = +1$, we have

$$\begin{aligned} \pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} &= \frac{1}{2} \ln \left| \left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha \right) \left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \beta \right) \right|, \\ &\quad + \left(\frac{\alpha + \beta}{2} + \frac{2\varepsilon\omega + h}{3h\varepsilon} \right) \frac{1}{\alpha - \beta} \ln \left| \frac{3h\varepsilon(u - \alpha) + \varepsilon\omega - h}{3h\varepsilon(u - \beta) + \varepsilon\omega - h} \right|. \end{aligned} \tag{46}$$

Case 2.4. $D_4 = 0, D_3 > 0, D_2 > 0$. We have

$$F(w) = (w - \alpha_1)^2 (w - \alpha_2)(w - \alpha_3), \tag{47}$$

where $\alpha_1, \alpha_2, \alpha_3$ are real numbers and $\alpha_2 > \alpha_3$. If$\epsilon = +1$, when $\alpha_1 > \alpha_2, w > \alpha_1$, we have

$$\begin{aligned} \pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} &= \ln \left(\sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2} + \sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3} \right)^2 - \frac{\alpha_1 + \frac{2\varepsilon\omega + h}{3h\varepsilon}}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \\ &\quad \times \ln \frac{\left[\sqrt{\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2 \right)(\alpha_1 - \alpha_3)} + \sqrt{(\alpha_1 - \alpha_2)\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3 \right)} \right]^2}{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_1}; \end{aligned} \tag{48}$$

when $\alpha_1 > \alpha_2, \alpha_1 > w > \alpha_2$, we have

$$\begin{aligned} \pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} &= \ln \left(\sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2} + \sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3} \right)^2 - \frac{\alpha_1 + \frac{2\varepsilon\omega + h}{3h\varepsilon}}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \\ &\quad \times \ln \frac{\left[\sqrt{\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2 \right)(\alpha_1 - \alpha_3)} + \sqrt{(\alpha_1 - \alpha_2)\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3 \right)} \right]^2}{\alpha_1 - u - \frac{\varepsilon\omega - h}{3h\varepsilon}}; \end{aligned} \tag{49}$$

when $\alpha_1 > \alpha_2, w < \alpha_3$, we have

$$\begin{aligned} \pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} &= \ln \left(\sqrt{\alpha_2 - u - \frac{\varepsilon\omega - h}{3h\varepsilon}} - \sqrt{\alpha_3 - u - \frac{\varepsilon\omega - h}{3h\varepsilon}} \right)^2 - \frac{\alpha_1 + \frac{2\varepsilon\omega + h}{3h\varepsilon}}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \\ &\quad \times \ln \frac{\left[\sqrt{\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2 \right)(\alpha_3 - \alpha_1)} - \sqrt{(\alpha_2 - \alpha_1)\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3 \right)} \right]^2}{\alpha_1 - u - \frac{\varepsilon\omega - h}{3h\varepsilon}}; \end{aligned} \tag{50}$$

when $\alpha_2 > \alpha_1 > \alpha_3, w > \alpha_2$, we have

$$\begin{aligned} \pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} &= \ln \left(\sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2} + \sqrt{u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3} \right)^2 + \frac{\alpha_1 + \frac{2\varepsilon\omega + h}{3h\varepsilon}}{\sqrt{(\alpha_2 - \alpha_1)(\alpha_1 - \alpha_3)}} \\ &\quad \times \arcsin \frac{\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_2 \right)(\alpha_1 - \alpha_3) + (\alpha_1 - \alpha_2)\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_3 \right)}{\left(u + \frac{\varepsilon\omega - h}{3h\varepsilon} - \alpha_1 \right)(\alpha_2 - \alpha_3)}; \end{aligned} \tag{51}$$

when $\alpha_2 > \alpha_1 > \alpha_3$, $w < \alpha_3$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left(\sqrt{\alpha_2-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}-\sqrt{\alpha_3-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}\right)^2+\frac{\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}}{\sqrt{(\alpha_2-\alpha_1)(\alpha_1-\alpha_3)}} \times \arcsin\frac{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_1-\alpha_3)+(\alpha_1-\alpha_2)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}{\left(\alpha_1-u-\frac{\varepsilon\omega-h}{3h\varepsilon}\right)(\alpha_2-\alpha_3)}; \tag{52}$$

when $\alpha_1 < \alpha_3$, $w > \alpha_2$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left(\sqrt{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2}+\sqrt{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3}\right)^2-\frac{\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \times \ln\frac{\left[\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_3-\alpha_1)}-\sqrt{(\alpha_2-\alpha_1)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}\right]^2}{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_1}; \tag{53}$$

when $\alpha_1 < \alpha_3$, $\alpha_3 > w > \alpha_1$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left(\sqrt{\alpha_2-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}-\sqrt{\alpha_3-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}\right)^2-\frac{\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \times \ln\frac{\left[\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_1-\alpha_3)}+\sqrt{(\alpha_1-\alpha_2)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}\right]^2}{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_1}; \tag{54}$$

when $\alpha_1 < \alpha_3$, $w < \alpha_1$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left(\sqrt{\alpha_2-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}-\sqrt{\alpha_3-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}\right)^2-\frac{\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \times \ln\frac{\left[\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_1-\alpha_3)}+\sqrt{(\alpha_1-\alpha_2)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}\right]^2}{\alpha_1-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}. \tag{55}$$

If $\epsilon = -1$, when $\alpha_1 > \alpha_2$, $\alpha_2 > w > \alpha_3$, we have

$$\pm\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \arcsin\frac{-2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)+\alpha_2+\alpha_3}{\alpha_2-\alpha_3}+\frac{\left(\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \times \arcsin\frac{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_1-\alpha_3)+(\alpha_1-\alpha_2)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}{\left(\alpha_1-u-\frac{\varepsilon\omega-h}{3h\varepsilon}\right)(\alpha_2-\alpha_3)}. \tag{56}$$

When $\alpha_2 > \alpha_1 > \alpha_3$, $\alpha_2 > w > \alpha_1$, we have

$$\pm\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \arcsin\frac{-2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)+\alpha_2+\alpha_3}{\alpha_2-\alpha_3}+\frac{\left(\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)}{\sqrt{(\alpha_2-\alpha_1)(\alpha_1-\alpha_3)}} \times \ln\frac{\left[\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_3-\alpha_1)}+\sqrt{(\alpha_2-\alpha_1)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}\right]^2}{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_1}. \tag{57}$$

When $\alpha_2 > \alpha_1 > \alpha_3$, $\alpha_1 > w > \alpha_3$, we have

$$\pm\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \arcsin\frac{-2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)+\alpha_2+\alpha_3}{\alpha_2-\alpha_3}+\frac{\left(\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)}{\sqrt{(\alpha_2-\alpha_1)(\alpha_1-\alpha_3)}} \times \ln\frac{\left[\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_3-\alpha_1)}+\sqrt{(\alpha_2-\alpha_1)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}\right]^2}{\alpha_1-u-\frac{\varepsilon\omega-h}{3h\varepsilon}}. \tag{58}$$

When $\alpha_1 < \alpha_3, \alpha_2 > w > \alpha_3$, we have

$$\pm\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \arcsin\frac{-2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)+\alpha_2+\alpha_3}{\alpha_2-\alpha_3} + \frac{\left(\alpha_1+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \times \arcsin\frac{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_2\right)(\alpha_1-\alpha_3)+(\alpha_1-\alpha_2)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_3\right)}{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha_1\right)(\alpha_2-\alpha_3)}. \tag{59}$$

Case 2.5. $D_4 = 0, D_3 = 0, D_2 > 0, E_2 = 0$. We have

$$F(w) = (w-\alpha)^3(w-\beta), \tag{60}$$

where α, β are real numbers. If $\epsilon = +1$, when $w > \alpha, w > \beta$, or $w < \alpha, w < \beta$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left|2\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha\right)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\beta\right)}+2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)-\alpha-\beta\right| - \frac{2(\alpha+\frac{2\varepsilon\omega+h}{3h\varepsilon})}{(\alpha-\beta)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha\right)}\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha\right)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\beta\right)}. \tag{61}$$

If $\epsilon = -1$, when $w > \alpha, w < \beta$, or $w < \alpha, w > \beta$, we have

$$\pm\frac{\sqrt{-\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \arcsin\frac{2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}\right)-\alpha-\beta}{|\alpha-\beta|} + \frac{2\left(\alpha+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)\sqrt{\left(\alpha-u-\frac{\varepsilon\omega-h}{3h\varepsilon}\right)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\beta\right)}}{(\alpha-\beta)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha\right)}. \tag{62}$$

Case 2.6. $D_4 = 0, D_2D_3 < 0$. Then we have

$$F(w) = (w-\alpha)^2[(w-l_1)^2+{s_1}^2], \tag{63}$$

where α, l_1 and s_1 are real numbers. If $\epsilon = +1$, we have

$$\pm\frac{\sqrt{\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \ln\left|2\sqrt{\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-l_1\right)^2+{s_1}^2}+2\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-l_1\right)\right| - \frac{\alpha+\frac{2\omega\varepsilon+h}{3h\varepsilon}}{\sqrt{(l_1-\alpha)^2+{s_1}^2}} \times \ln\left|\frac{2\left\{\sqrt{[(l_1-\alpha)^2+s_1^2]\left[\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-l_1\right)^2+s_1^2\right]}+(\alpha-l_1)\left(u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha\right)+[(l_1-\alpha)^2+s_1^2]\right\}}{u+\frac{\varepsilon\omega-h}{3h\varepsilon}-\alpha}\right|. \tag{64}$$

Case 2.7. $D_4 > 0, D_3 > 0, D_2 > 0$. Then we have

$$F(w) = (w-\alpha_1)(w-\alpha_2)(w-\alpha_3)(w-\alpha_4), \tag{65}$$

where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are real numbers, and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. By assuming that

$$w = \frac{a\sin^2\varphi+b}{c\sin^2\varphi+d}, \quad \left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right), \tag{66}$$

when the parameters satisfy one of the following four conditions, we have

$$\pm\frac{\sqrt{\epsilon\varepsilon}(\xi-\xi_0)}{\sqrt{2}h} = \left(\frac{a}{c}+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)\frac{2\delta}{\gamma}F(\varphi,k) - \frac{2\delta^2}{cd\gamma}\Pi(\varphi,\rho,k). \tag{67}$$

Condition 1. $\epsilon = +1$, $w > \alpha_1$ or $w < \alpha_4$, $k^2 = \frac{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}$, $a = \alpha_2(\alpha_1-\alpha_4)$,

$$b = -\alpha_1(\alpha_2-\alpha_4), \qquad c = \alpha_1-\alpha_4, \qquad d = -(\alpha_2-\alpha_4), \qquad \rho = \frac{c}{d},$$

$$\delta = (\alpha_1-\alpha_2)(\alpha_1-\alpha_4)(\alpha_2-\alpha_4),$$

$$\frac{\delta}{\gamma} = \frac{1}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}, \qquad \Pi(\varphi,\rho,k) = \int_0^{\sin\varphi}\frac{\mathrm{d}\eta}{(1+\rho\eta^2)\sqrt{(1-\eta^2)(1-k^2\eta^2)}}. \tag{68}$$

Condition 2. $\epsilon=+1, \alpha_2>w>\alpha_3, k^2=\frac{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}, a=\alpha_4(\alpha_2-\alpha_3)$,

$$b=-\alpha_3(\alpha_2-\alpha_4),\qquad c=\alpha_2-\alpha_3,\qquad d=-(\alpha_2-\alpha_4),\qquad \rho=\frac{c}{d},$$
$$\delta=(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4),\quad \frac{\delta}{\gamma}=\frac{1}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}. \tag{69}$$

Condition 3. $\epsilon=-1, \alpha_1>w>\alpha_2, k^2=\frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}, a=\alpha_3(\alpha_1-\alpha_2)$

$$b=-\alpha_2(\alpha_1-\alpha_3),\qquad c=\alpha_1-\alpha_2,\qquad d=-(\alpha_1-\alpha_3),\qquad \rho=\frac{c}{d},$$
$$\delta=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3),\quad \frac{\delta}{\gamma}=\frac{1}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}. \tag{70}$$

Condition 4. $\epsilon=-1, \alpha_3>w>\alpha_4, k^2=\frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}, a=\alpha_1(\alpha_3-\alpha_4)$,

$$b=-\alpha_4(\alpha_1-\alpha_3),\qquad c=\alpha_3-\alpha_4,\qquad d=\alpha_1-\alpha_3,\qquad \rho=\frac{c}{d},$$
$$\delta=(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_3-\alpha_4),\qquad \frac{\delta}{\gamma}=\frac{1}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}. \tag{71}$$

Case 2.8. $D_4<0, D_2D_3\geqslant 0$. Then we have

$$F(w)=(w-\alpha_1)(w-\alpha_2)[(w-l_1)^2+{s_1}^2], \tag{72}$$

where α_1, α_2, l_1, and s_1 are real numbers, and $\alpha_1>\alpha_2, s_1>0$. By assuming that

$$w=\frac{a\cos\varphi+b}{c\cos\varphi+d},\quad (0\leqslant\varphi\leqslant\pi), \tag{73}$$

we have

$$\pm\frac{\sqrt{\epsilon\varepsilon}(\xi-\xi_0)}{\sqrt{2}h}=-\frac{\delta}{\gamma}\left(\frac{a}{c}+\frac{2\varepsilon\omega+h}{3h\varepsilon}\right)F(\varphi,k)+\frac{\delta^2}{c\gamma(d^2-c^2)}[d\Pi(\varphi,\rho,k)-cD_4(\varphi,\rho,k)], \tag{74}$$

where

$$k_1=A\pm\sqrt{A^2+1},\quad A=\frac{s_1^2+(\alpha_1-l_1)(\alpha_2-l_1)}{s_1(\alpha_1-\alpha_2)},\qquad a=\frac{1}{2}(\alpha_1+\alpha_2)c-\frac{1}{2}(\alpha_1-\alpha_2)d,$$
$$b=\frac{1}{2}(\alpha_1+\alpha_2)d-\frac{1}{2}(\alpha_1-\alpha_2)c,\qquad c=\alpha_1-l_1-\frac{s_1}{k_1},\qquad d=\alpha_1-l_1+s_1k_1,\qquad k^2=\frac{1}{1+k_1^2},$$
$$\delta=ad-bc=\frac{1}{2}(\alpha_1-\alpha_2)(c^2-d^2),\qquad \gamma=[(\alpha_1-l_1)^2+s_1^2]\sqrt{\epsilon(c^2-d^2)[(\alpha_2-l_1)^2+s_1^2]},$$
$$\frac{\delta}{\gamma}=\frac{-2kk_1}{\sqrt{-2\epsilon s_1k_1(\alpha_1-\alpha_2)}},\qquad \rho=\frac{c^2}{d^2-c^2},$$
$$D_4(\varphi,\rho,k)=\int_0^{\varphi}\frac{\cos\psi}{(1+\rho\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}\mathrm{d}\psi,\quad (\rho>-k^2),$$
$$D_4(\varphi,\rho,k)=\frac{1}{2\sqrt{-k^2-\rho}}\ln\frac{\sqrt{1-k^2\sin^2\varphi}+\sqrt{-k^2-\rho}\sin\varphi}{\sqrt{1-k^2\sin^2\varphi}-\sqrt{-k^2-\rho}\sin\varphi},\quad \left(\rho<-k^2, 0\leqslant\sin^2\varphi<\frac{-1}{\rho}\right)$$
$$D_4(\varphi,\rho,k)=\frac{\sin\psi}{\sqrt{1-k^2\sin^2\psi}},\quad (\rho=-k^2), \tag{75}$$

where we choose k_1 such that $\epsilon k_1<0$. *Case* 2.9. $D_4>0, D_2D_3\leqslant 0$. Then we have

$$F(w)=[(w-l_1)^2+{s_1}^2][(w-l_2)^2+{s_2}^2], \tag{76}$$

where l_1, l_2, s_1 and s_2 are real numbers, and $s_1>s_2>0$. By assuming that

$$w=\frac{a\tan\varphi+b}{c\tan\varphi+d}, \tag{77}$$

if $\epsilon = +1$, we have

$$\pm \frac{\sqrt{\varepsilon}(\xi - \xi_0)}{\sqrt{2}h} = \left(\frac{ac+bd}{c^2+d^2} + \frac{2\varepsilon\omega + h}{3h\varepsilon}\right)\frac{\delta}{\gamma}F(\varphi, k) - \frac{c\delta^2}{\gamma d(c^2+d^2)}\Pi(\varphi, \rho, k)$$
$$+ \frac{\delta^2}{2\gamma\sqrt{(c^2+d^2)(c^2+d^2-k^2d^2)}} \ln \frac{\sqrt{c^2+d^2}\sqrt{1-k^2\sin^2\varphi} + \sqrt{c^2+d^2-k^2d^2}}{\sqrt{c^2+d^2}\sqrt{1-k^2\sin^2\varphi} - \sqrt{c^2+d^2-k^2d^2}}, \tag{78}$$

where

$$k^2 = \frac{k_1^2 - 1}{k_1^2}, \quad k_1 = A + \sqrt{A^2-1},\ A = \frac{(l_1-l_2)^2 + s_1^2 + s_2^2}{2s_1s_2},$$
$$a = l_1c + s_1d, \qquad b = -s_1c + l_1d, \qquad c = -s_1 + \frac{1}{k_1}s_2, \qquad d = l_1 - l_2, \qquad \rho = -\frac{c^2+d^2}{d^2},$$
$$\delta = ad - bc = s_1(c^2+d^2), \qquad \gamma = s_1s_2\sqrt{(c^2+d^2)(k_1^2+d^2)}. \tag{79}$$

3. Conclusion and discussion

In the paper, by the complete discrimination system for polynomial method, we obtained the classifications of all single traveling wave solutions to the Vakhnenko equation and a generalized Vakhnenko equation. If we take $f(u)$ as some other functions such as $f(u) = u^3$, we can also give the corresponding classification. However, if we take $f(u)$ as some forms such as $f(u) = \exp u$, it is not easy to reduce the corresponding integral to the form in terms of a polynomial. Therefore, it is difficult to solve out the integrals so the solutions must be expressed by the integral forms or approximate forms.

In Ref. [19], Abazari gave several exact solutions to the following Vakhnenko–Parkes equation by the $\frac{G'}{G}$ method,

$$uu_{xxt} - u_xu_{xt} + u^2u_t = 0. \tag{80}$$

However, by Liu's method, we can easily give the classification of its single traveling wave solutions. Indeed, under the transformation (11) in Ref. [19], Eq. (80) can be reduced to the ODE

$$u'' - \frac{1}{u}(u')^2 + \frac{u^2}{3k^2} = 0, \tag{81}$$

whose general solution is given by the formula (7) in our paper,

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}u}{u\sqrt{c - \frac{u}{3k^2}}}, \tag{82}$$

where ξ_0 and c are two integral constants. Letting $c - \frac{u}{3k^2} = v^2$, we have

$$\pm \frac{\xi - \xi_0}{2} = \int \frac{\mathrm{d}v}{v^2 - c}. \tag{83}$$

According to three cases of $c > 0$, $c < 0$, $c = 0$, we can easily give all solutions of v and corresponding u. These solutions of u are just the solutions (18)–(20) in Ref. [19].

Acknowledgments

I would like to thank the referees for their helpful suggestions.

References

[1] C.S. Liu, Chin. Phys. 16 (2007) 1832.
[2] C.S. Liu, Commun. Theor. Phys. 45 (2006) 991.
[3] C.S. Liu, Chin. Phys. 14 (2005) 1710.
[4] C.S. Liu, Commun. Theor. Phys. 49 (2008) 153.
[5] C.S. Liu, Commun. Theor. Phys. 48 (2007) 601.
[6] C.S. Liu, Comput. Phys. Commun. 181 (2010) 317.
[7] D.S. Wang, H.B. Li, J. Math. Anal. Appl. 343 (2008) 273.
[8] S. Yang, Modern Phys. Lett. B 24 (2010) 363.
[9] V.A. Vakhnenko, J. Phys. A: Math. Gen. 25 (1992) 4181.
[10] V.O. Vakhnenko, Ukr. J. Phys. 42 (1997) 104.
[11] E.J. Parkes, J. Phys. A: Math. Gen. 26 (1993) 6469.

[12] W.A. Li, H. Chen, G.C. Zhang, Chin. Phys. 18 (2009) 400.
[13] Y.Y. Wu, C. Wang, S.J. Liao, Chaos Solitons Fractals 23 (2005) 1733.
[14] Z.B. Li, The Traveling Wave Solutions to Nonlinear Mathematics Physical Equations, Science Press, Beijing, 2007, p.18.
[15] J.Q. MO, Chin. Phys. B 18 (2009) 4608.
[16] A.J. Morrison, E.J. Parkes, V.O. Vakhnenko, Nonlinearity 12 (1999) 1427.
[17] V.O. Vakhnenko, E.J. Parkes, Rep. NAS Ukr. 8 (2006) 88.
[18] V.O. Vakhnenko, E.J. Parkes, Proceedinds of the Fifth International Conference Symmetry in Nonlinear Mathematical Physics, Kyiv. vol. 50, 2004, p. 493.
[19] R. Abazari, Comput. Fluids 39 (2010) 1957.